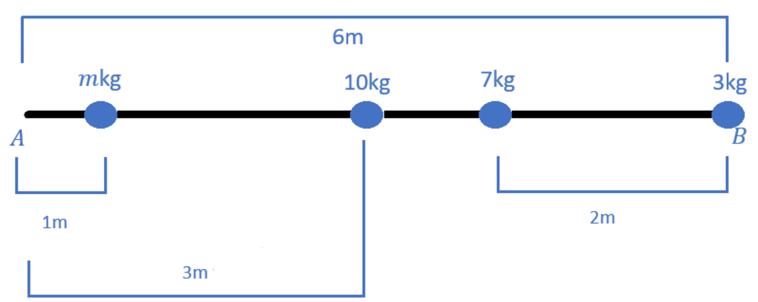
## Centres of Mass of Composite Bodies

Composite bodies are comprised of different shapes, and so to find their centre of mass, the centres of mass of all the component parts are taken, and a weighted average is calculated. For a composite body of $n$ components, each of mass $m_i$ and centre of mass $\binom{\bar{x}_i}{\bar{y}_i}$, the centre of mass of the composite body, $\binom{\bar{x}}{\bar{y}}$, is given by

$$M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = m_1\begin{pmatrix}\bar{x}_1\\ \bar{y}_1\end{pmatrix} + \cdots + m_n\begin{pmatrix}\bar{x}_n\\ \bar{y}_n\end{pmatrix}, \qquad M = \sum_{i=1}^{n} m_i .$$

**Example 1:** A composite body is comprised of a uniform rod $AB$ of mass 10kg and length 6m with a mass of 3kg attached at $B$, a mass of 7kg attached 2m from $B$, and a mass of $m$kg attached 1m from $A$. Given that the centre of mass is 2m from $A$, find $m$.

Begin by drawing a diagram of the composite body. Since the rod is uniform, its centre of mass is at its midpoint, and so it acts as a mass of 10kg placed 3m from either end.




Find the total mass of the system and set up the equation to find the centre of mass of the system, remembering to write all distances relative to $A$. Use this to find an equation for $m$, and solve this.

$$M = 10 + 7 + 3 + m = 20 + m$$

2m from $B$ is $(6-2) =$ 4m from $A$.

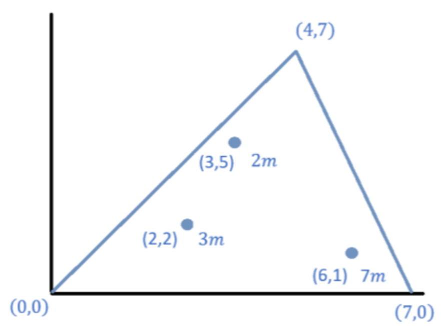
$$2 = \frac{1}{20+m}\left((m\cdot 1) + (10\cdot 3) + (7\cdot 4) + (3\cdot 6)\right) = \frac{1}{20+m}(m+76)$$

$$\therefore 2(20+m) = 40 + 2m = m + 76 \Rightarrow m = 76 - 40 = 36$$

$\therefore$ The required mass is 36kg.

**Example 2:** A composite body consists of a uniform triangular lamina with mass $4m$ kg and vertices at $(0,0)$, $(7,0)$ and $(4,7)$, with masses of $3m$ kg, $2m$ kg and $7m$ kg attached at points $(2,2)$, $(3,5)$ and $(6,1)$ respectively. Find the centre of mass of the composite body.

Begin by drawing a diagram of the composite body and finding its mass. Then, find the centre of mass of the triangular lamina using that it's at the mean of the three vertices. The centre of mass of each particle is at its coordinates since particles are concentrated points of mass. Use the weighted average formula to find the position of the composite body's centre of mass.




The total mass is

$$M = 4m + 2m + 3m + 7m = 16m$$

Centre of mass of the triangular lamina $\binom{\overline{x_T}}{\overline{y_T}}$:

$$\overline{x_T} = \frac{1}{3}(0+4+7) = \frac{11}{3}$$

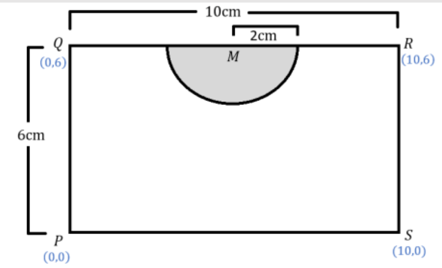
$$\overline{y_T} = \frac{1}{3}(0+0+7) = \frac{7}{3}$$

$$\therefore \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{16m}\left(4m\begin{pmatrix}\frac{11}{3}\\ \frac{7}{3}\end{pmatrix} + 3m\begin{pmatrix}2\\ 2\end{pmatrix} + 2m\begin{pmatrix}3\\ 5\end{pmatrix} + 7m\begin{pmatrix}6\\ 1\end{pmatrix}\right)$$

$$= \frac{1}{16}\begin{pmatrix}\frac{206}{3}\\ \frac{97}{3}\end{pmatrix} = \begin{pmatrix}\frac{103}{24}\\ \frac{97}{48}\end{pmatrix}$$

**Example 3:** A composite body consists of a rectangular uniform lamina $PQRS$, where $PQ = 6$cm, $QR = 10$cm, with a semi-circle of radius 2cm cut out of it, where the centre of the base of the semicircle is at the midpoint $M$ of $QR$. Find the centre of mass of this composite body.

Draw a diagram of the lamina, including the measurements, setting up edges $PS$ as a horizontal axis and $PQ$ as a vertical axis, with $P$ acting as the origin. Using these axes, coordinates can be given to the vertices. Find the coordinates of $M$ given that it is midway between $Q$ and $R$.




$M$ is midway between points $Q$ and $R$, and therefore has coordinates

$$M = (5,6)$$

Since the laminae are uniform, and aren't given masses, work with their areas as their masses are directly proportional to them. Begin by finding the areas of both laminae, and then the centre of mass of the rectangle. This will be at the average of the four vertices.

A semi-circle is a sector with angle $\pi$ radians, and so use the formula for the centre of mas of a sector with angle $2\alpha$ to find the centre of mass of the semi-circle. This will lie directly below $M$ since the axis of symmetry of the semi-circle is through its midpoint.

Area of the rectangular lamina $(A_R)$ and of the semi-circle $(A_c)$:

$$A_R = 10\cdot 6 = 60\text{cm}^2,\ A_c = \tfrac{1}{2}(\pi\cdot 2^2) = 2\pi\text{cm}^2 \ \therefore \text{ total area is } 60 - 2\pi \text{ cm}^2$$

Centre of mass of the rectangular lamina, $\binom{\overline{x_R}}{\overline{y_R}}$:

$$\overline{x_R} = \frac{1}{4}(0+0+10+10) = 5, \qquad \overline{y_R} = \frac{1}{4}(0+6+0+6) = 3$$

The centre of mass of a sector of a circle of radius $r$ with an angle of $2\alpha$ radians is $\frac{2r\sin(\alpha)}{3\alpha}$ from the centre of the circle, along the axis of symmetry. In this case, $2\alpha = \pi$ $\therefore$ the centre of mass is

$$\frac{2(2)\sin\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2}} = \frac{8}{3\pi}$$

vertically below the point $M$. Therefore, the coordinates of the semi-circle's centre of mass are

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}5\\ 6-\frac{8}{3\pi}\end{pmatrix}$$

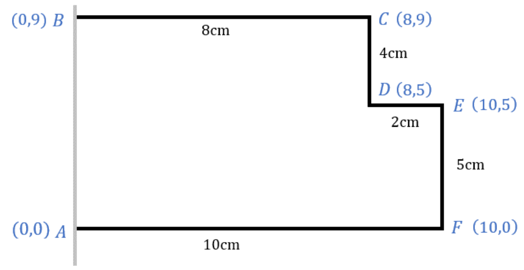
Find the centre of mass of the composite body by taking the centre of mass of the semi-circle multiplied by its area away from the product of the rectangle's area with its centre of mass. Finally, divide by the area of the composite shape.

The centre of mass of the composite body is

$$(60-2\pi)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 60\begin{pmatrix}5\\ 3\end{pmatrix} - 2\pi\begin{pmatrix}5\\ 6-\frac{8}{3\pi}\end{pmatrix} \Rightarrow \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{60-2\pi}\begin{pmatrix}300-10\pi\\ \frac{556}{3}-12\pi\end{pmatrix}$$

**Example 4:** A 29cm piece of uniform wire is bent into the following shape $ABCDEF$. Treating the $A$ as the origin of a coordinate system, find the distance of the centre of mass from $A$.

As the wire is uniform, the mass of each piece is directly proportional to its length, and each piece's centre of mass is at its midpoint. Break the composite shape down into each segment of wire and find the coordinates of each piece's midpoint.




Multiply the coordinates of each piece of wire's centre of mass with its length and sum over the whole shape. Divide by the total length of wire to find the centre of mass of the composite shape.

Total length is 29cm.

$$29\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 10\begin{pmatrix}5\\ 0\end{pmatrix} + 5\begin{pmatrix}10\\ \frac{5}{2}\end{pmatrix} + 2\begin{pmatrix}9\\ 5\end{pmatrix} + 4\begin{pmatrix}8\\ 7\end{pmatrix} + 8\begin{pmatrix}4\\ 9\end{pmatrix} = \begin{pmatrix}182\\ 122.5\end{pmatrix}$$

$$\therefore \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{182}{29}\\ \frac{245}{58}\end{pmatrix}$$

The distance from $A$ to $\begin{pmatrix}\frac{182}{29}\\ \frac{245}{58}\end{pmatrix}$ is $\left|\begin{pmatrix}\frac{182}{29}\\ \frac{245}{58}\end{pmatrix}\right| = \sqrt{\left(\frac{182}{29}\right)^2 + \left(\frac{245}{58}\right)^2} = 7.57$cm, since $A$ is the origin.